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STATIONARY RENEWAL PROCESSES

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Abstract—Let $(t_n = z_0 + \cdots + z_n)_{\mathbb{N}}$ be renewal instants sequence. Suppose that the random variables $(z_n)_{\mathbb{N}}$ are independent and the corresponding renewal process is stationary (in weak sense) with a finite positive intensity. Then there exists an integer $d \geq 1$ such that the sequence $(t_{nd})_{n\geq 0}$ forms a stationary *recurrent* renewal process and $z_m = 0$ almost surely if $m \neq nd$, $\quad \forall n \in \mathbb{N}$.

Let $(z_n)_{\mathbb{N}}$ be a sequence of non-negative r.v. (random variables). The measure N on $[\mathbb{R}, B(\mathbb{R})]$ with values in $\mathbb{N}$ such that

$$N(A) = \sum_{i\geq 0} \mathbb{I}_A(t_i), \qquad \forall\, A \in B(\mathbb{R}),$$

where $t_i = z_0 + \cdots + z_i$, $i \geq 0$, is a *renewal process.* This process is called *stationary (in weak sense)* if the distribution of the r.v.

$$[N(A_1 + t),\, N(A_2 + t)]$$

is independent on $t \geq 0$ for all A_1 and A_2 from $B(\mathbb{R}_+)$. In particular, putting $N(t) = N([0,t))$, $\forall t \geq 0$ we have $\mathsf{E}N(x + y) = \mathsf{E}N(x) + \mathsf{E}N(y)$, $\quad \forall\, x \geq 0,\, y \geq 0$ and consequently $\mathsf{E}N(t) = \lambda t$, $\quad \forall\, t \geq 0$ with $0 \leq \lambda \leq +\infty$. The number λ is called intensity of the renewal process N.

Statement. *Suppose that*

1) *the r.v.* $z_0, z_1, \ldots$ *are independent*
2) *the renewal process* N *is stationary in weak sense*
3) $0 < \lambda < \infty$.

Then $\exists\, d \in \mathbb{N}_+$ *such that*

1) $z_m = 0$ a.s., $\quad \forall\, m \notin d\,\mathbb{N} = \{0, d, 2d, \ldots\}$
2) *the r.v.* $(z_{nd})_{n\geq 0}$ *are independent and the r.v.* $(z_{nd})_{n\geq 1}$ *are i.i.d. (independent identically distributed)*
3) $P(z_0 < x) = a \int_0^x P(z_d > u)\, du$, $\quad \forall\, x \geq 0$ *with* $\lambda = ad$.

Lemma 1.

$$A_k(t) = \sum_{n\geq 0} P_n(T) A_{n+k}(t), \qquad \forall\, k \geq 1,\, T \geq 0, \tag{1}$$

where

$$A_k(t) = P(z_k < t),\; P_n(T) = P\{N(T) = n\}.$$

Proof of Lemma 1. The proof of Lemma 1 is divided into items.

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ITEM 1. Denote the right part of Equation (1) by $R(T)$. It is enough to verify that $R(T)$ is independent on $T \geq 0$ [because $R(0) = A_k(t)$]. It follows from

$$R(T) = \lim_{n\to\infty} \sum_{m\in \mathbb{I}} [P(N(A_{mn}+T) < k,\ N(C_{mn}+T) \geq k+1) \\ - P(N(B_{mn}+T) < k,\ N(C_{mn}+T) \geq k+1)], \quad (2)$$

where

$$A_{mn} = [0, m2^{-n}),\ B_{mn} = [0, (m+1)2^{-n}),\ C_{mn} = [0, m2^{-n}+t).$$

Now it is sufficient to verify (2).

ITEM 2. For any r.v. ξ and η and real number x put

$$A = (\xi - \eta < x),\ A_n = \bigcup_{m\in \mathbb{I}} \{\eta \in [m2^{-n},\ (m+1)2^{-n}),\ \xi < m2^{-n} + x\}.$$

Then $A_n \uparrow A$. If $\eta \geq 0$ then $\mathbb{I}$ can be replaced by $\mathbf{N}$. Show it.

2.1 $A_n \subset A$. Indeed, $\omega \in A_n \Rightarrow \exists m \in \mathbb{I}$ such that $\eta(\omega) \in [m2^{-n}, (m+1)2^{-n}), \xi(\omega) < m2^{-n} + x \Rightarrow \xi(\omega) < \eta(\omega) + x \Rightarrow \omega \in A$.

2.2 $A_n \subset A_{n+1}$. Indeed, $\omega \in A_n \Rightarrow \exists m \in \mathbb{I}$ such that $\eta(\omega) \in [m2^{-n}, (m+1)2^{-n}), \xi(\omega) < m2^{-n} + x \Rightarrow \eta(\omega) \in [2m \cdot 2^{-(n+1)}, (2m+1)2^{-(n+1)}) \cup [(2m+1)2^{-(n+1)}, (2m+2)2^{-(n+1)})$, $\xi(\omega) < 2m \cdot 2^{-(n+1)} + x < (2m+1)2^{-(n+1)} + x \Rightarrow \omega \in A_{n+1}$.

2.3 $A \subset \bigcup A_n$. Indeed, $\omega \in A \Rightarrow \exists n \in \mathbf{N}$ such that $\xi(\omega) - \eta(\omega) < x - 2^{-n}$. Then $\exists m \in \mathbb{I}$ such that $\eta(\omega) \in [m2^{-n}, (m+1)2^{-n})$ and, in addition, $\xi(\omega) = \eta(\omega) + [\xi(\omega) - \eta(\omega)] < (m+1)2^{-n} + x - 2^{-n} = m2^{-n} + x$. It gives $\omega \in A_n$.

ITEM 3. Using the independence the r.v. $(z_n)_{\mathbf{N}}$ and the notation $N(t) = \sup\{i+1 : t_i < t\} = N([0,t))$ with $\sup\{\emptyset\} = 0$, we have

$$R(T) = P\{\bigcup_{i\geq 0} (N(T) = i, t_{i+k} - t_{i+k-1} < t)\}.$$

Note that $N(T) < \infty$ a.s. (since $\lambda < \infty$). Putting $\xi = t_{i+k} - T$, $\eta = t_{i+k-1} - T$ and using Item 2 and the relations

$$(t_i < t) = (N(t) \geq i+1), \qquad (t_i \geq t) = (N(t) < i+1),$$

we have

$$\begin{aligned} R(T) &= \lim_{n\to\infty} P\Big\{ \bigcup_{i,m\in\mathbf{N}} (N(T) = i, t_{i+k-1} - T \in [m2^{-n}, (m+1)2^{-n}), t_{i+k} - T < m2^{-n} + t)\Big\} \\ &= \lim P\Big\{\bigcup \Big(N(T) = i, t_{i+k-1} \geq m2^{-n} + T, t_{i+k-1} \\ &\quad < (m+1)2^{-n} + T, t_{i+k} < m2^{-n} + t + T\Big)\Big\} \\ &= \lim P\Big\{\bigcup\Big(N(T) = i, N(m2^{-n}+T) < i+k, N\big((m+1)2^{-n}+T\big) \geq i+k, \\ &\quad N(m2^{-n}+t+T) \geq i+k+1\Big)\Big\} \\ &= \lim P\Big\{\bigcup\Big(N(T) = i, N(A_{mn}+T) < k, N(B_{mn}+T) \geq k, N(C_{mn}+T) \geq k+1\Big)\Big\} \\ &= \lim P\left\{\bigcup_{m\in\mathbf{N}} \Big(N(A_{mn}+T) < k, N(B_{mn}+T) \geq k, N(C_{mn}+T) \geq k+1\Big)\right\}. \end{aligned}$$

Note that the events

$$E_m = \{N(A_{mn}+T) < k,\ N(B_{mn}+T) \geq k,\ N(C_{mn}+T) \geq k+1\}, \qquad m \in \mathbb{I},$$

are disjoint (in pairs) since for $i \geq 1$

$$E_m \bigcap E_{m+i} \subset \{N(B_{mn}+T) \geq k,\ N(A_{m+i,n}+T) < k\} = \emptyset$$

due to $B_{mn} \subset A_{m+i,n}$. It remains to use

$$\begin{aligned} E_m &= \{N(A_{mn}+T) < k,\ N(C_{mn}+T) \geq k+1\} \setminus \\ &\quad \{N(A_{mn}+T) < k,\ N(C_{mn}+T) \geq k+1,\ N(B_{mn}+T) < k\} \\ &= \{N(A_{mn}+T) < k,\ N(C_{mn}+T) \geq k+1\} \setminus \\ &\quad \{N(B_{mn}+T) < k,\ N(C_{mn}+T) \geq k+1\} \end{aligned}$$

and $A_{mn} \subset B_{mn}$.

REMARK. Due to Equation (2), $R(T)$ is independent on $T > 0$ if $P\{N(A+T) = k\}$ and $P\{N(A+T) = k,\ N(B+T) = 0\}$ are independent on T for all $k \in \mathbb{N}$, $A = [0,a)$, $B = [a,b)$, $b > a > 0$. This condition of stationarity is more weak than indicated in the one above.

LEMMA 2. *Every bounded solution of the system (with respect to $u_0, u_1, \dots$)*

$$u_k = \sum_{i \geq 0} p_i u_{k+i}, \qquad k \geq 0, \tag{3}$$

where $p_0 < 1$, $p_i \geq 0$, $\forall i \geq 0$, $\sum_{i \geq 0} p_i = 1$, is periodic, i.e., $\exists$ an integer d, $1 \leq d \leq \min\{i : p_i \neq 0,\ i \geq 1\}$ such that $u_{k+d} = u_k$, $\forall k \geq 0$.

It follows from the proposition (see [1]): Let $p = (p_n)_{\mathbb{I}}$ be a sequence of real numbers with $\sum_{n \in \mathbb{I}} |p_n| < \infty$. Suppose that the continuous function

$$p(\theta) = \sum_{n \in \mathbb{I}} p_n e^{in\theta}, \qquad -\pi \leq \theta \leq \pi,$$

has a finite number of zeros $\theta_1, \dots, \theta_d$ $(-\pi \leq \theta_1 < \dots < \theta_d < \pi)$. Then every bounded solution $u = (u_m)_{\mathbb{I}}$, $|u_n| \leq c$, of the system

$$\sum_{n \in \mathbb{I}} p_n u_{m+n} = 0, \qquad m \in \mathbb{I},$$

has a form

$$u_m = \sum_{1}^{d} \alpha_s \exp\{\mathrm{i} m \theta_s\}, \qquad m \in \mathbb{I},$$

where $a_1, \dots, a_d$ are some complex numbers. Note that the proof of this proposition uses the following theorem: every minimal closed primary ideal of Winer ring coincides with maximal ideal.

LEMMA 3. *There exists an integer $d \geq 1$ such that $A_{nd+r} = A_r$, $\forall$ integers $n \geq 0$ and $1 \leq r \leq d$. In addition, if $d > 1$ then*

$$A_1(t) = \dots = A_{d-1}(t) = 1, \qquad \forall t > 0.$$

PROOF. We have $P_0(T) < 1$, $\forall T > 0$ since $\mathbb{E}N(t) = \lambda t$, $\forall t \geq 0$ and $0 < \lambda < \infty$. Define an integer $d \geq 1$ by following condition: $\exists T > 0$ such that $P_d(T) \neq 0$ and if $d > 1$ then

$$P_1(t) = \dots = P_{d-1}(t) = 0, \qquad \forall t > 0.$$

Thanks to Lemma 2, $\forall t > 0$ the solution $u_k = A_{k+1}(t)$, $k \geq 0$, of the system (3) with $p_i = P_i(T)$, $i \geq 0$, is periodic with a period $d_t, 1 \leq d_t \leq d$.

If $d = 1$ then $A_1(t) = A_2(t) = \ldots \ \forall t > 0$, hence the statement of this lemma is true. Consider now the case $d > 1$. As

$$P_1(t) = A_0(t) - (A_0 \star A_1)(t) = 0, \qquad \forall t > 0,$$

we have $A_1(t) = 1, \quad \forall t > 0$. Using the formula

$$P_k(t) = P(t_{k-1} < t) - P(t_k < t) = (A_0 \star \cdots \star A_{k-1})(t) - (A_0 \star \cdots \star A_k)(t), \ k \geq 1,$$

we obtain by induction $A_k(t) = 1, \quad \forall t > 0, 1 \leq k < d$. If $d_t = d$ then $u_k = u_{d+k} = u_{2d+k} = \cdots$ for $k = 0, 1, \ldots, d-1$ and, hence, $A_r(t) = A_{nd+r}(t)$ for $n \geq 0$, $r = 1, \ldots, d$. If now $d_t < d$ then

$$A_{nd_t+r}(t) = A_r(t),$$

for all $n \geq 0$, $r = 1, \ldots, d_t < d$, but $A_r(s) = 1, \quad \forall s > 0$, for $r = 1, \ldots, d-1$. Hence, $A_m(t) = 1, \quad \forall m \geq 1$. It gives $A_{nd+r}(t) = A_r(t) = 1, \quad \forall n \geq 0, 1 \leq r \leq d$; it means that the statement of Lemma 3 is true in all cases.

Now the main *Statement* follows from Lemma 3 and from $\mathsf{E}N(t) = \lambda t, \quad \forall t > 0$ with $0 < \lambda < \infty$ if we use the renewal equation for the recurrent renewal process $(z_{nd})_{n \in \mathbb{N}}$.

REFERENCES

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